

# DISCRETE VARIANT OF THE INITIAL INTEGRATION METHOD AND ITS APPLICATION TO THE SOLUTION OF BIHARMONIC EQUATIONS

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*Let's consider the boundary value problem for the following double harmonic equation:*

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = -f(x, y), -1 < x, y < 1, \quad (1)$$

$$u(-1, y) = 0, u(1, y) = 0,$$

$$u(x, -1) = 0, u(x, 1) = 0.$$

$$\frac{\partial u}{\partial x}(-1, y) = 0, \frac{\partial u}{\partial x}(1, y) = 0, \quad (2)$$

$$\frac{\partial u}{\partial y}(x, -1) = 0, \frac{\partial u}{\partial y}(x, 1) = 0.$$

Choosing a test function (determining solution) for problems (1)-(2)

$$u_e^{(1)}(x, y) = (1 - x^2)^2 (1 - y^2)^2 e^{A \sin x \sin y}, \quad (3)$$

$$u_e^{(2)}(x, y) = (1 - x^2)^2 (1 - y^2)^2 e^{A(x+y)}.$$

To determine the form of the  $f(x, y)$  function from formula (1), we substitute solution (3) into equation (1) and will have the following appearance:

$$\begin{aligned} f(x, y) = & -e^{A \sin x \sin y} ((y^2 - 1)^2 (A \sin y (3(x^2 - 1)^2 \sin^2 x + \sin x (-6A^2 (x^2 - 1)^2 \times \\ & \times \sin^2 y \cos^2 x - 48Ax(x^2 - 1) \sin y \cos x + (x^4 - 74x^2 + 25)) + A \sin y \cos^2 x \times \\ & \times (A^2 (x^2 - 1)^2 \sin^2 y \cos^2 x + 16Ax(x^2 - 1) \sin y \cos x - 4(x^4 - 20x^2 + 7)) + \\ & + 16x(7 - x^2) \cos x) + 2 \cdot ((A(y^2 - 1)^2 (x^2 - 1) (-2A(x^2 - 1) \cos 2y \cos^2 x - \\ & - \sin x \sin y (x^2 - 1) + 8x \cos x \sin y) + A(y^2 - 1)^2 \sin x \sin y (A(x^2 - 1)^2 \sin y \times \\ & \times (-\cos^2 x \sin y + \sin x) + 8Ax(1 - x^2) \cos x \sin y - 12x^2 + 4) - A(y^2 - 1) \sin x \times \\ & \times \cos y (A \sin x (y^2 - 1) \cos y + 8y) (-A^2 (x^2 - 1)^2 \cos^2 x \sin^2 y + A(x^2 - 1)^2 \sin x \sin y - \\ & - 8Ax(x^2 - 1) \cos x \sin y - 12x^2 + 4) - 4(3y^2 - 1) (-A^2 (x^2 - 1)^2 \cos^2 x \sin^2 y + \\ & + A(x^2 - 1)^2 \sin x \sin y - 8Ax(x^2 - 1) \cos x \sin y - 12x^2 + 4) - A(y^2 - 1) (x^2 - 1) \cos y \times \\ & \times (2A(y^2 - 1) \sin x \cos y + 8y) (-2A(x^2 - 1) \cos^2 x \sin y + (x^2 - 1) \sin x - 8x \cos x))) + \\ & + (x^2 - 1)^2 (A \sin x (3(y^2 - 1)^2 \sin^2 y + \sin y (-6A^2 (y^2 - 1)^2 \sin^2 x \cos^2 y - 48Ay \times \\ & \times (y^2 - 1) \sin x \cos y + (y^4 - 74y^2 + 25)) + A \sin x \cos^2 y (A^2 (y^2 - 1)^2 \sin^2 x \cos^4 y + \\ & + 16Ay(y^2 - 1) \sin x \cos^3 y - 4(y^4 - 20y^2 + 7)) + 16y(7 - y^2) \cos y) + 24)). \end{aligned}$$

To numerically model the differential problem (1)-(2), we first use the discrete variant of the integration method. We search for the approximate solution  $u_a(x, y)$  and the right-hand function  $f(x, y)$  in the form of a series of finite Chebyshev polynomials of the first kind:

$$u_a(x, y) = \sum_{i=0}^N \sum_{j=0}^M a_{ij} T_i(x) T_j(y), \quad f(x, y) = \sum_{i=0}^N \sum_{j=0}^M g_{ij} T_i(x) T_j(y), \quad (4)$$

here,  $a_{ij}$  and  $g_{ij}$  are unknowns, and the bar code in the sums means that if the indices  $i$  or  $j$  are equal to 0, then they are multiplied by  $\frac{1}{2}$ , and if they are  $i = j = 0$ , then they are multiplied by  $\frac{1}{4}$ .

In this case, we enter the following lines to calculate the derivatives:

$$\begin{aligned} \frac{\partial^4 u_a}{\partial x^4} &= \sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(4,x)} T_i(x) T_j(y), \\ \frac{\partial^4 u_a}{\partial x^2 \partial y^2} &= \sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(2x,2y)} T_i(x) T_j(y), \\ \frac{\partial^4 u_a}{\partial y^4} &= \sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(4,y)} T_i(x) T_j(y). \end{aligned} \quad (5)$$

Based on formulas (4)-(5) and equation (1), we form this equation [1-3]:

$$\sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(4,x)} T_i(x) T_j(y) + \sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(2x,2y)} T_i(x) T_j(y) + \sum_{i=0}^N \sum_{j=0}^M a_{ij}^{(4,y)} T_i(x) T_j(y) = - \sum_{i=0}^N \sum_{j=0}^M g_{ij} T_i(x) T_j(y),$$

or by equating the coefficients in front of Chebyshev polynomials of the same order, we form this equation:

$$\begin{aligned} a_{ij}^{(4,x)} + a_{ij}^{(2x,2y)} + a_{ij}^{(4,y)} &= -g_{ij}, \\ i &= 0, 1, 2, \dots, N, \\ j &= 0, 1, 2, \dots, M. \end{aligned}$$

By using a discrete formula of the integration method[1-2]:

$$a_i^{(k-1)x} = \frac{a_{i-1}^{kx} - a_{i+1}^{kx}}{2i}, \quad (6)$$

both sides of this equation are integrated four times. Then, from the expression of boundary conditions (2) in terms of rows, we form this algebraic system (7):



$$\left[ \begin{aligned}
 &K_1(j^2 - 1)((j+1)(j+2)(j+3)a_{i,j-4} + (j-1)(j-2)(j-3)a_{i,j+4}) - (K_1(j^2 - 1) \cdot \\
 &\cdot (j+1)(j+2)(j+3) + Z_1)a_{i,j-2} - (K_1(j^2 - 1)(j-1)(j-2)(j-3) + Z_2)a_{i,j+2} + \\
 &+ (Z_1 + Z_2)a_{i,j} + K_2(j^2 - 9)(i+1)(j+1)a_{i-2,j-2} - 2K_2j(j^2 - 9)(i+1)a_{i-2,j} + \\
 &+ K_2(j^2 - 9)(i+1)(j-1)a_{i-2,j+2} + K_2(j^2 - 9)(i-1)(j+1)a_{i+2,j-2} - 2K_2j \cdot \\
 &\cdot (j^2 - 9)(i-1)a_{i+2,j} + K_2(j^2 - 9)(i-1)(j-1)a_{i+2,j+2} + 16j^2(j^2 - 1)^2(j^2 - 4) \cdot \\
 &\cdot (j^2 - 9)(C_1T_1 - C_2T_2 + C_3T_3) = -C_1(i-1)j((j+1)(j+2)(j+3)(j^2 - 1)g_{i-4,j-4} - \\
 &- 4(j^2 - 1)(j^2 - 4)(j+3)g_{i-4,j-2} + 6j(j^2 - 1)(j^2 - 9)g_{i-4,j} - 4(j^2 - 1)(j^2 - 4) \cdot \\
 &\cdot (j-3)g_{i-4,j+2} + (j-1)(j-2)(j-3)(j^2 - 1)g_{i-4,j+4}) + j(2c_1(i-2) + C_2(i+1)) \cdot \\
 &\cdot ((j+1)(j+2)(j+3)(j^2 - 1)g_{i-2,j-4} - 4(j^2 - 1)(j^2 - 4)(j+3)g_{i-2,j-2} + \\
 &+ 6j(j^2 - 1)(j^2 - 9)g_{i-2,j} - 4(j^2 - 1)(j^2 - 4)(j-3)g_{i-2,j+2} + (j-1)(j-2)(j-3) \cdot \\
 &\cdot (j^2 - 1)g_{i-2,j+4}) - j(C_1(i-3) + 2C_2i + C_3(i+3))((j+1)(j+2)(j+3)(j^2 - 1)g_{i,j-4} - \\
 &- 4(j^2 - 1)(j^2 - 4)(j+3)g_{i,j-2} + 6j(j^2 - 1)(j^2 - 9)g_{i,j} - 4(j^2 - 1)(j^2 - 4) \cdot \\
 &\cdot (j-3)g_{i,j+2} + (j-1)(j-2)(j-3)(j^2 - 1)g_{i,j+4}) + j(C_2(i-1) + 2C_3(i+2)) \cdot \\
 &\cdot ((j+1)(j+2)(j+3)(j^2 - 1)g_{i+2,j-4} - 4(j^2 - 1)(j^2 - 4)(j+3)g_{i+2,j-2} + \\
 &+ 6j(j^2 - 1)(j^2 - 9)g_{i+2,j} - 4(j^2 - 1)(j^2 - 4)(j-3)g_{i+2,j+2} + (j-1)(j-2)(j-3) \cdot \\
 &\cdot (j^2 - 1)g_{i+2,j+4}) - C_3(i+1)j((j+1)(j+2)(j+3)(j^2 - 1)g_{i+4,j-4} - 4(j^2 - 1)(j^2 - 4) \cdot \\
 &\cdot (j+3)g_{i+4,j-2} + 6j(j^2 - 1)(j^2 - 9)g_{i+4,j} - 4(j^2 - 1)(j^2 - 4)(j-3)g_{i+4,j+2} + (j-1) \cdot \\
 &\cdot (j-2)(j-3)(j^2 - 1)g_{i+4,j+4}), \\
 &\left. \begin{aligned}
 &\frac{1}{2}a_{i,0} + a_{i,2} + a_{i,4} + \dots + a_{i,2M} = 0, \\
 &a_{i,1} + a_{i,3} + a_{i,5} + \dots + a_{i,2M-1} = 0, \\
 &4a_{i,2} + 16a_{i,4} + 36a_{i,6} + \dots + (2M)^2 a_{i,2M} = 0, \\
 &a_{i,1} + 9a_{i,3} + 25a_{i,5} + \dots + (2M-1)^2 a_{i,2M-1} = 0,
 \end{aligned} \right\} (i = \overline{0, N}), \\
 &\left. \begin{aligned}
 &\frac{1}{2}a_{0,j} + a_{2,j} + a_{4,j} + \dots + a_{2N,j} = 0, \\
 &a_{1,j} + a_{3,j} + \dots + a_{2N-1,j} = 0, \\
 &4a_{2,j} + 16a_{4,j} + 36a_{6,j} + \dots + (2N)^2 a_{2N,j} = 0, \\
 &a_{1,j} + 9a_{3,j} + 25a_{5,j} + \dots + (2N-1)^2 a_{2N-1,j} = 0,
 \end{aligned} \right\} (j = \overline{4, M}),
 \end{aligned} \right\} \tag{7}$$

here





$$\begin{aligned}
 K_1 &= 16i^2 j(i^2 - 1)^2(i^2 - 4)(i^2 - 9), \\
 K_2 &= 32ij(i^2 - 1)(i^2 - 4)(i^2 - 9)(j^2 - 1)(j^2 - 4), \\
 T_1 &= (i - 1)a_{i-4,j} - 2(i - 2)a_{i-2,j} + (i - 3)a_{i,j}, \\
 T_2 &= (i + 1)a_{i-2,j} - 2ia_{i,j} + (i - 1)a_{i+2,j}, \\
 T_3 &= (i + 3)a_{i,j} - 2(i + 2)a_{i+2,j} + (i + 1)a_{i+4,j}, \\
 Z_1 &= (3K_1(j - 1)(j + 2) + 2K_2i)(j^2 - 9)(j + 1), \\
 Z_2 &= (3K_1(j - 2)(j + 1) + 2K_2i)(j^2 - 9)(j - 1), \\
 C_1 &= i(i + 1)^2(i + 2)(i + 3), \\
 C_2 &= 2i(i^2 - 4)(i^2 - 9), \\
 C_3 &= i(i - 1)^2(i - 2)(i - 3).
 \end{aligned}$$

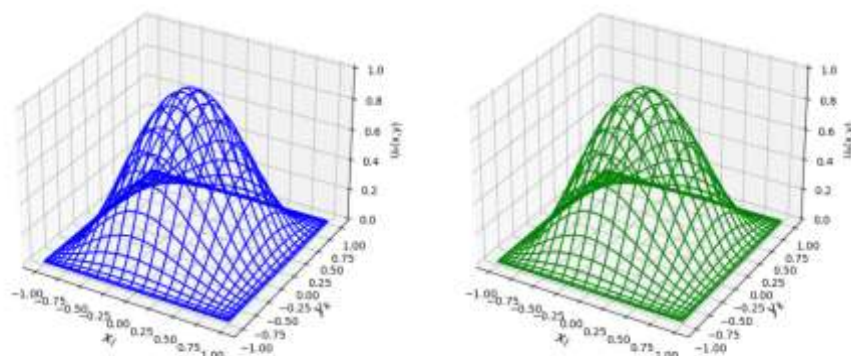
In this system,  $(N + 1) \cdot (M + 1)$  unknowns  $a_{ij} (i = 0, 1, \dots, N; j = 0, 1, \dots, M)$  are determined.

From system (7), the coefficients  $a_i$  are obtained and substituted into the formula for the approximate solution:

$$u_a(x, y) = \sum_{i=0}^N \sum_{j=0}^M a_{ij} T_i(x) T_j(y)$$

The values of the approximate solution at the nodal points  $(x_i, y_k)$  are computed and compared with the exact solution  $u_e(x, y)$  of the given problem.

The dynamics of the exact and approximate solutions for the function  $u_e(x, y)$  are illustrated in Fig. 1 for the case  $A = 1, N = M = 50$ .



**Fig. 1.** a) exact solution  $u_e$       б) approximate solution  $u_a$





## Literature

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