



VECTORS: FUNDAMENTAL CONCEPTS, PROPERTIES, AND APPLICATIONS IN MATHEMATICS AND SCIENCE

Mingniyozova Ugulshod Kodirovna

Mathematics Teacher

Academic Lyceum of Termez University of Engineering and Agrotechnologies

Phone: +998 91 978 07 10

Abstract. Vectors are among the most important concepts in mathematics, physics, engineering, and computer science. Unlike scalar quantities, vectors possess both magnitude and direction, making them essential for describing physical phenomena such as force, velocity, acceleration, and displacement. This paper examines the fundamental properties of vectors, vector operations, and their applications in various scientific disciplines. The study highlights the significance of vector algebra in solving geometric and real-world problems and demonstrates how vectors provide a mathematical framework for modeling complex systems.

Keywords: vector, magnitude, direction, vector algebra, mathematics, geometry, physics.

Introduction

Mathematics serves as the foundation of modern science and technology, and vectors constitute one of its most powerful tools. The concept of a vector emerged from the need to represent quantities that require both magnitude and direction. Today, vectors are widely used in geometry, mechanics, navigation, computer graphics, and data analysis.

The study of vectors enables students and researchers to understand spatial relationships and solve practical problems involving motion and forces. This paper explores the theoretical foundations of vectors and discusses their applications in different fields.

Main Body

Concept of a Vector

A vector is a mathematical quantity characterized by both magnitude and direction. It is usually represented by an arrow whose length corresponds to the magnitude and whose orientation indicates the direction.

A vector is commonly denoted as **a**, **b**, or (\vec{a}) . If a vector begins at point A and ends at point B, it is represented as (\overrightarrow{AB}) .

Basic Types of Vectors

Zero Vector: A vector with zero magnitude.

Unit Vector: A vector whose magnitude equals one.

Equal Vectors: Vectors with the same magnitude and direction.

Opposite Vectors: Vectors with equal magnitudes but opposite directions.

Collinear Vectors: Vectors lying on the same or parallel lines.

Coplanar Vectors: Vectors lying in the same plane.

A vector is a quantity that has both magnitude (length) and direction. Unlike a scalar quantity, a vector cannot be described by a number alone; its direction must also be specified.

Notation: \vec{a} , \overrightarrow{AB} , or in boldface as **a**, **AB**.

Graphically, a vector is represented by an arrow with an **initial point** (A) and a **terminal point** (B).

1.1 Basic Concepts

- **Magnitude (Length) of a Vector:** $|\vec{a}|$ or $|\mathbf{a}|$ — a positive number or zero.
- **Zero Vector:** $|\vec{0}| = 0$ — its magnitude is zero, and its direction is undefined.
- **Unit Vector:** $|\vec{e}| = 1$ — a vector whose magnitude is equal to 1.
- **Equal Vectors:** Vectors that have the same magnitude and direction, regardless of their initial points.
- **Opposite Vectors:** Vectors that have the same magnitude but opposite directions $(-\vec{a})$.
- **Collinear Vectors:** Vectors that lie on the same straight line or on parallel lines.
- **Coplanar Vectors:** Vectors that lie in the same plane.

Coordinate Representation of Vectors

In a coordinate system, a vector is expressed through its components:

$$\vec{a} = (a_1, a_2) \text{ in a plane (2D)}$$

— in a plane (2D)

$$\vec{a} = (a_1, a_2, a_3) \text{ in space (3D)}$$

— in space (3D)

Calculating the Magnitude of a Vector

For a vector in a plane (2D):

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

For a vector in space (3D):

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example

Given the vector

$$\vec{a} = (3, 4)$$

its magnitude is

$$|\vec{a}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Therefore, the magnitude (length) of the vector \vec{a} is 5.

Operations on Vectors

2.1 Vector Addition

The addition of two vectors is performed according to the following rules:

Triangle Rule: Place vector \vec{b} at the terminal point (head) of vector \vec{a} . The resultant vector is drawn from the initial point of \vec{a} to the terminal point of \vec{b} .

Parallelogram Rule: Draw both vectors from the same initial point and construct a parallelogram using the vectors as adjacent sides. The diagonal of the parallelogram represents the sum of the two vectors.

If

$$\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2), \vec{a} = (a_1, a_2), \quad \vec{b} = (b_1, b_2), a = (a_1, a_2), b = (b_1, b_2),$$

then their sum is

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2) \quad \vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2) \quad a + b = (a_1 + b_1, a_2 + b_2)$$

Example:

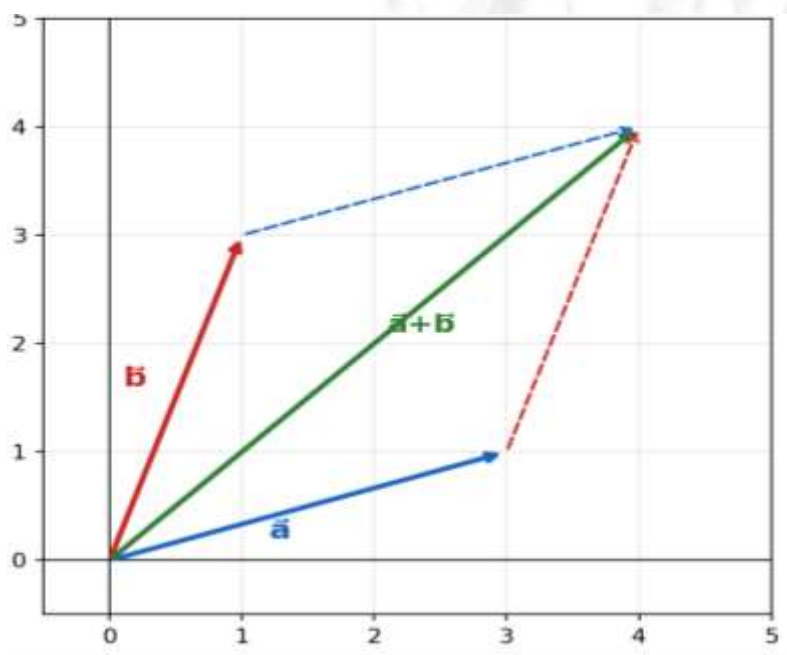
If

$$\vec{a} = (3, 1), \vec{b} = (1, 3), \vec{a} = (3, 1), \quad \vec{b} = (1, 3), a = (3, 1), b = (1, 3),$$

then

$$\vec{a} + \vec{b} = (3 + 1, 1 + 3) = (4, 4). \quad \vec{a} + \vec{b} = (3 + 1, 1 + 3) = (4, 4). \quad a + b = (3 + 1, 1 + 3) = (4, 4).$$

Thus, the resultant vector is **(4, 4)**.



Properties of Vector Addition

For vectors

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2) \quad \vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2) \quad a + b = (a_1 + b_1, a_2 + b_2)$$

Vector addition has the following properties:

- **Commutative Property:**

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad \vec{a} + \vec{b} = \vec{b} + \vec{a} \quad a + b = b + a$$



- **Associative Property:**

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

- **Identity Element (Zero Vector):**

$$\vec{a} + \vec{0} = \vec{a} \quad \vec{a} + \vec{0} = \vec{a}$$

2.2 Multiplication of a Vector by a Scalar

When a scalar λ is multiplied by a vector \vec{a} , the result is:

$$\lambda \cdot \vec{a} = (\lambda a_1, \lambda a_2) \quad \lambda \cdot \vec{a} = (\lambda a_1, \lambda a_2)$$

Properties:

- If $\lambda > 0$, the direction of the vector remains unchanged, while its magnitude is multiplied by λ .
- If $\lambda < 0$, the direction of the vector is reversed.
- If $\lambda = 0$, the result is the **zero vector**.

2.3 Dot Product (Scalar Product)

The **dot product** of two vectors is a scalar quantity obtained by multiplying their corresponding components and adding the results.

For vectors

$$\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2), \quad \vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 \quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

In terms of the angle θ between the vectors:

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \quad \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

The dot product is used to:

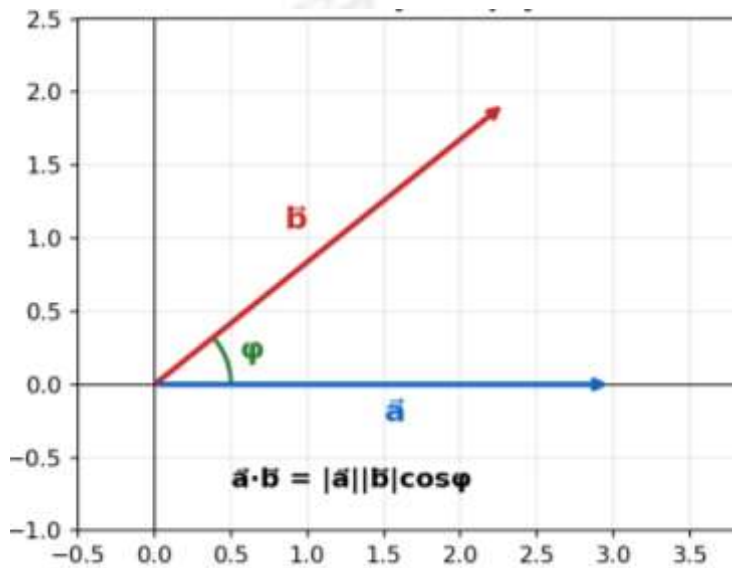
- Find the angle between two vectors.
- Determine whether vectors are perpendicular.
- Calculate projections in geometry and physics.

If

$$\vec{a} \cdot \vec{b} = 0, \quad \vec{a} \cdot \vec{b} = 0, \quad \vec{a} \cdot \vec{b} = 0,$$

then the vectors are **orthogonal (perpendicular)**.





Dot Product (Scalar Product)

The **dot product** is a numerical value that depends on the angle between two vectors:

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \varphi$$

Using coordinates:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

where φ is the angle between the vectors ($0^\circ \leq \varphi \leq 180^\circ$).

- If $\vec{a} \cdot \vec{b} = 0$, the vectors are **perpendicular** ($\varphi = 90^\circ$).
- If $\vec{a} \cdot \vec{b} > 0$, the angle is **acute**.
- If $\vec{a} \cdot \vec{b} < 0$, the angle is **obtuse**.

3. Cross Product and Applications

3.1 Vector (Cross) Product

For two vectors in three-dimensional space, the **cross product** is defined. The result is a new vector that is perpendicular to both original vectors:

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \varphi \vec{n}$$

where \vec{n} is a unit vector perpendicular to both \vec{a} and \vec{b} , determined by the **right-hand rule**.

Using coordinates:

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

Properties:

- If $\vec{a} \times \vec{b} = \vec{0}$, the vectors are **collinear** (parallel).



- $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \varphi$ represents the **area of the parallelogram** formed by vectors \vec{a} and \vec{b} .

3.2 Calculating the Angle Between Two Vectors

The angle between two vectors can be found using:

$$\cos \varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \Rightarrow \varphi = \arccos \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

Example

Let

$$\vec{a} = (1, 0, 0), \vec{b} = (0, 1, 0), \vec{a} = (1, 0, 0), \quad \vec{b} = (0, 1, 0), \vec{a} = (1, 0, 0), \vec{b} = (0, 1, 0).$$

Then

$$\vec{a} \cdot \vec{b} = 0, |\vec{a}| = 1, |\vec{b}| = 1, \vec{a} \cdot \vec{b} = 0, \quad |\vec{a}| = 1, \quad |\vec{b}| = 1.$$

Therefore,

$$\cos \varphi = 0 \Rightarrow \varphi = 90^\circ$$

and

$$\varphi = 90^\circ, \varphi = 90^\circ, \varphi = 90^\circ.$$

Thus, the vectors are **perpendicular**.

3.3 Projection

The projection of vector \vec{a} onto the direction of vector \vec{b} is given by:

$$\text{pr}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \vec{b}$$

The projection measures how much of vector \vec{a} lies in the direction of vector \vec{b} .

3.4 Practical Applications

Vectors are widely used in physics, engineering, and technology:

- **Force, velocity, and acceleration** are vector quantities.
- **Work** is calculated using the dot product:

$$W = \vec{F} \cdot \vec{s} = |\vec{F}| |\vec{s}| \cos \varphi \Rightarrow W = F s \cos \varphi$$

- **Torque and rotational motion** are analyzed using the cross product.
- **Computer graphics** use vectors for transformations, rotations, and object positioning.
- **Robotics and navigation systems** rely heavily on vector calculations for movement and orientation.

3.5 Midpoint and Section Formulas

For points





$A(x_1, y_1)$ and $B(x_2, y_2)$, $A(x_1, y_1) \rightarrow B(x_2, y_2)$, $A(x_1, y_1)$ and $B(x_2, y_2)$,

the vector \overrightarrow{AB} is

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$

Conclusion

Vectors play a fundamental role in mathematics and its applications. Their ability to describe both magnitude and direction makes them indispensable in science, engineering, and technology. Through vector operations and algebraic techniques, complex real-world problems can be modeled and solved effectively. As scientific and technological advancements continue, the importance of vector analysis will remain central to modern education and research.

References

1. Anton, H. *Elementary Linear Algebra*. Wiley.
2. Lay, D. C. *Linear Algebra and Its Applications*. Pearson.
3. Strang, G. *Introduction to Linear Algebra*. Wellesley-Cambridge Press.
4. Kreyszig, E. *Advanced Engineering Mathematics*. Wiley.
5. Stewart, J. *Calculus: Early Transcendentals*. Cengage Learning.

