MATRITSANING MODULI BO‘YICHA ENG KATTA XOS SON VA UNGA MOS VEKTORLARINI TOPISH

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Annotatsiya: *Ushbu maqolada matritsaning moduli (absolyut qiymati) bo‘yicha eng katta xos sonini (eigenvalue) aniqlash va unga mos xos vektorlarni topish masalasi yoritilgan. Xususan, bu qiymatlarning chiziqli algebra va amaliy matematikadagi o‘rni, dinamik tizimlarni tahlil qilish, differensial tenglamalarni yechish hamda fizikaviy va muhandislik modellarida qo‘llanilishi ko‘rib chiqilgan. Matritsaning xos sonlarini topishda kuchaytirish (power method) usuli asosida iteratsion yondashuvlar tahlil qilinadi va algoritmik yechimlar misollar orqali ko‘rsatib beriladi. Tadqiqot natijalari yordamida real va kompleks xos sonlar uchun moduli bo‘yicha eng katta qiymatni aniqlashning samarali usullari yoritilgan. Maqola talabalarga, tadqiqotchilarga va muhandislarga matritsalar nazariyasini amaliyotda qo‘llashda yordam berishni maqsad qiladi.*

Kalit so‘zlar: *matritsa, xos son, xos vektor, eng katta modulli xos son, kuchaytirish usuli, iteratsion yondashuv, chiziqli algebra, sonli misollar, kompleks sonlar, matritsalar nazariyasi*

Abstract: *This article addresses the problem of determining the eigenvalue of a matrix with the largest absolute value (modulus) and its corresponding eigenvectors. Special attention is given to the role of these quantities in linear algebra and applied mathematics, including applications in dynamic system analysis, solving differential equations, and modeling in physics and engineering. Iterative approaches such as the power method are discussed, and algorithmic solutions are illustrated through specific examples. The results highlight efficient techniques for finding the dominant eigenvalue in both real and complex cases. The article aims to support students, researchers, and engineers in applying matrix theory to practical problems.*

Keywords: *matrix, eigenvalue, eigenvector, dominant eigenvalue, power method, iterative approach, linear algebra, numerical examples, complex numbers, matrix theory.*

Аннотация: *В данной статье рассматривается задача нахождения собственного числа матрицы с наибольшим модулем и соответствующих ему собственных векторов. Особое внимание уделяется роли этих величин в линейной алгебре и прикладной математике, а также в анализе динамических систем, решении дифференциальных уравнений и моделировании физических и инженерных процессов. Изложены итерационные подходы к нахождению собственных чисел, в частности, метод степеней. Алгоритмы представлены на конкретных примерах, демонстрирующих эффективность определения наибольшего по модулю собственного числа как для действительных, так и для комплексных случаев. Цель статьи — способствовать практическому применению теории матриц студентами, исследователями и инженерами.*

Ключевые слова: *матрица, собственное значение, собственный вектор, собственное значение с наибольшим модулем, метод степеней, итерационный подход, линейная алгебра, численные примеры, комплексные числа, теория матриц.*

Faraz qilaylik matritsaning barcha xos vektorlari chiziqli erkli bo‘lsin. Bu holda matritsa oddiy strukturaga ega deyiladi.

1-hol. A matritsaning xos qiymatlari quyidagi tengsizlikni qanoatlantirsin

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n| \quad (1)$$

Biz λ_1 ning taqribiy qiymatini topish usulini ko‘rsatamiz. Ixtiyoriy noldan farqli $y^{(0)}$ vektor olib, uni A matritsaning xos vektorlari $x^{(i)}$ lar bo‘yicha yoyamiz:

$$y^{(0)} = b_1x^{(1)} + b_2x^{(2)} + \ldots + b_nx^{(n)}$$

bu yerda b_j lar o‘zgarmas sonlar. $y^{(0)}$ vektor ustida A^k matritsa yordamida almashtirish bajaramiz:

$$y^{(k)} = A^k y^{(0)} = \sum_{j=1}^{n} b_j A^k x^{(j)} = \sum_{j=1}^{n} b_j \lambda^k x^{(j)}.$$

Bundan $Ax^{(j)} = \lambda_j x^{(j)}$ ligini e'tiborga olsak,

$$y^{(k)} = \sum_{j=1}^{n} b_j \lambda_i^k x^{(j)} \quad (2)$$

bo‘ladi.

Endi n o‘lchovli vektorlar fazosida $l_1, l_2, \ldots, l_n$ bazis olamiz. Bu bazisda $x^{(j)}$ vektorni yoyib yozamiz:

$$x^{(j)} = \sum_{i=1}^{n} x_{ij} l_i. \quad (3)$$

Endi (2) dan (3) ga asosan

$$y^{(k)} = \sum_{j=1}^{n} b_i \lambda_j^k \sum_{i=1}^{n} x_{ij} l_i$$

ni hosil qilamiz. Bunda yig'ish tartibini o'zgartirib,

$$y^{(k)} = \sum_{i=1}^{n} l_i \sum_{j=1}^{n} b_j \lambda_j^k x_{ij} \quad (4)$$

ga ega bo'lamiz. Ichki summa l_i ning koeffitsiyenti, demak u $y^{(k)}$ vektorning i - koordinatasidir. Bundan quyidagini yoza olamiz:

$$y_i^{(k)} = \sum_{j=1}^{n} b_j \lambda_j^k x_{ij} \quad (5)$$

Xuddi shuningdek,

$$y_1^{(k+1)} = \sum_{j=1}^{n} b_j \lambda_j^{k+1} x_i \quad (6)$$

(6) ni (5) ga nisbatini olib,

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \frac{b_1 x_{i1} \lambda_1^{k+1} + b_2 x_{i2} \lambda_2^{k+1} + \dots + b_n x_{in} \lambda_n^{k+1}}{b_1 x_{i1} \lambda_1^k + b_2 x_{i2} \lambda_2^k + \dots + b_n x_{in} \lambda_n^k} \quad (7)$$

ga ega bo'lamiz.

Endi $b_1 x_{i1} \neq 0$ deylik, bunga erishish uchun $y^{(0)}$ vektorni va $l_1, l_2, \dots, l_n$ bazisni kerakli ravishda tanlash kerak.

(7) ni quyidagicha

$$\frac{y_l^{(k+1)}}{y_i^{(k)}} = \lambda_1 \frac{1 + \sum_{j=2}^{n} \frac{b_j x_{ij}}{b_1 x_{i1}} \left(\frac{\lambda_j}{\lambda_1}\right)^{k+1}}{1 + \sum_{j=2}^{n} \frac{b_j x_{ij}}{b_1 x_{i1}} \left(\frac{\lambda_j}{\lambda_1}\right)^{k}}$$

yozamiz. Bu yerdan $k \to \infty$ da (1) ga ko'ra

$$\lim_{k\to\infty} \frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1, i = 1,2, \dots, n$$

kelib chiqadi. Demak, yetarlicha katta k lar uchun

$$\lambda_1 \approx \frac{y_i^{(k+1)}}{y_i^{(k)}}, i = 1,2, \dots, n \quad (8)$$

deb olishimiz mumkin.

Aniqlangan λ_1 ga mos xos vektor sifatida $u^{(k)}$ ni olish mumkin. Haqiqatan, (2) ga ko'ra

$$y^{(k)} = b_1\lambda_j^k \left[x^{(1)} + \sum_{j=2}^{n} \frac{b_j}{b_1}\left(\frac{\lambda_j}{\lambda_1}\right)^k x^{(j)} \right]_j^7$$

bo‘ladi. Yetarlicha katta k lar uchun

$$y^{(k)} \cong b_1\lambda_1^k x^{(1)}$$

taqribiy tenglikka ega bo‘lamiz. $y^{(k)}$ xos vektor $x^{(1)}$ dan sonli ko‘paytuvchiga farq qilyapti, demak, $y\lambda_1$ xos songa mos keladigan xos vektordir. U matritsaning λ_1 xos soniga mos keluvchi $x^{(1)}$ xos vektorining yo‘nalishiga yaqin bo‘ladi. (8) ning o‘ng tomoni i=1,2….n lar uchun berilgan aniqlikda bir xil bo‘lishligi λ_1 va $x^{(1)}$ ga yaqinlashganlik darajasini anglatadi.

2-hol. A matritsa xos sonining moduli bo‘yicha eng kattasi karrali bo‘lsin, ya'ni $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_s$. Quyidagi

$$|\lambda_1| > |\lambda_{s+1}| \geq |\lambda_{s+2}| \geq \dots \geq |\lambda_n| \quad (9)$$

téngsizlik bajarilsin. Bu holda (7) tenglik

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \frac{(b_1x_{i1} + b_2x_{i2} + \dots + b_sx_{is})\lambda_1^{k+1} + b_{s+1}x_{is+1}\lambda_{s+1}^{k+1} + \dots + b_nx_{in}\lambda_n^{k+1}}{(b_1x_{i1} + b_2x_{i2} + \dots + b_sx_{is})\lambda_1^{k} + b_{s+1}x_{is+1}\lambda_{s+1}^{k} + \dots + b_nx_{in}\lambda_n^{k}}$$

ko‘rinishga ega bo‘ladi. Bunda $b_1x_{i1} + b_2x_{i2} + \dots + b_sx_{is} \neq 0$ deb faraz qilamiz va (10) ni quyidagi ko‘rinishda yozamiz:

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1 \frac{1 + \frac{1}{b_1x_{i1} + b_2x_{i2} + \dots + b_sx_{is}} \sum_{j=s+1}^{n} b_jx_{ij}\left(\frac{\lambda_j}{\lambda_1}\right)^{k+1}}{1 + \frac{1}{b_1x_{i1} + b_2x_{i2} + \dots + b_sx_{is}} \sum_{j=s+1}^{n} b_jx_{ij}\left(\frac{\lambda_j}{\lambda_1}\right)^{k}} \quad (10)$$

Bundan $k \to \infty$ da (9)ga ko‘ra

$$\lim_{k\to\infty} \frac{y_1^{(k+1)}}{y_i^{(k)}} = \lambda_1,\ i = 1,2,\dots,n$$

kelib chiqadi. Demak, yetarlicha katta k lar uchun

$$\lambda_1 \approx \frac{y_i^{(k+1)}}{y_i^{(k)}},\ i = 1,2,\dots,n$$

deb olishimiz mumkin. Bu esa (8) taqribiy tenglikning o‘zginasi, λ_1 ga mos vektor deb, 1-holdagidek $y^{(k)}$ ni olishimiz mumkin. Boshlang‘ich $y^{(0)}$ vektorni boshqacha olsak, boshqa xos, vektorni topish mumkin.

3-hol. A matritsaning xos sonlari quyidagi shartlarni qanoatlantirsin:

$$\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_r = -\lambda_{r+1} = -\lambda_{r+2} = \dots = -\lambda_{r+p} \quad (11)$$

va

$$|\lambda_1| = |\lambda_{r+p}| > |\lambda_{r+p+1}| \geq \dots \geq |\lambda_n|. \quad (12)$$

Bu holda yuqoridagi jarayondan foydalanilmaydi, chunki (5) quyidagi koʻrinishga ega boʻlib,

$$y_i^{(k)} = \sum_{j=1}^{r} b_j x_{ij} \lambda_1^k + \sum_{j=r+1}^{r+p} b_j x_{ij} (-1)^k \lambda_1^k + \sum_{j=r+p+1}^{n} b_j x_{ij} \lambda_j^k,$$

birinchi va ikkinchi summada λ_1 ning tartibi bir xil, lekin k ning oʻzgarishi bilan ikkinchi summa oʻz ishorasini oʻzgartiradi. Shuning uchun

$$\frac{y_i^{(k+1)}}{y_i^{(k)}}$$

nisbat $k \to \infty$ da limitga ega boʻlmaydi. Bu holda $y_i^{(2k)}$ va $y_i^{(2k+2)}$ yoki $y_i^{(2k-1)}$ va $y_i^{(2k+1)}$ dan foydalanib, λ_1^2 ni topish mumkin:

yoki

$$\frac{y_i^{(2k+2)}}{y_i^{(2k)}} \approx \lambda_1, \; i = 1,2, \dots, n$$

$$\frac{y_i^{(2k+1)}}{y_i^{(2k-1)}} \approx \lambda_1, \; i = 1,2, \dots, n.$$

A matritsaning λ_1 va $-\lambda_1$ xos sonlariga mos keladigan vektorlari esa mos ravishda $y^{(k+1)} + \lambda_1 y^{(k)}$ va $y^{(k+1)} - \lambda_1 y^{(k)}$ boʻladi. Haqiqatan ham, misol uchun $y^{(k+1)} + \lambda_1 y^{(k)}$ vektorni (11) ni e'tiborga olgan holda (2) ga ko^t ra quyidagicha yozamiz:

$$y^{(k+1)} + \lambda_1 y^{(k)} = \lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + (-1)^{k+1} \lambda_1^{k+1} \sum_{j=r+1}^{r+p} b_j x^{(j)} + \sum_{j=r+p+1}^{n} b_j \lambda_j^{k+1} x^{(j)} +$$

$$+ \lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + (-1)^{k+1} \lambda_1^{k} \sum_{j=r+1}^{r+p} b_j x^{(j)} + \lambda_1 \sum_{j=r+p+1}^{n} b_j \lambda_j^{k} x^{(j)} =$$

$$= 2\lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + \sum_{j=r+p+1}^{n} b_j (\lambda_1 + \lambda_j) \lambda_j^k x^{(j)}$$

Bundan esa, (12)ni nazarga olib

$$y^{(k+1)} + \lambda_1 y^{(k)} = \lambda_1^{k+1} \left(\sum_{j=1}^{r} 2b_j x^{(j)} + \sum_{j=r+p+1}^{n} (\lambda_1^2 + \lambda_1 \lambda_j) \left(\frac{\lambda_j}{\lambda_1} \right)^k b_j x^{(j)} \right)$$

ga ega boʻlamiz. Demak, yetarlicha katta k uchun

$$y^{(k+1)} + \lambda_1 y^{(k)} \cong \lambda_1^{k+1} \sum_{j=1}^{r} 2b_j x^{(j)}$$

boʻladi. Xuddi shuningdek,

$$y^{(k+1)} - \lambda_1 y^{(k)} \cong (-\lambda_1)^{k+1} \sum_{j=r+1}^{r+p} 2b_1 x^{(j)}$$

ekanligi ko‘rsatiladi.

Agar r va p yoki ulaming birortasi birdan katta bo‘lsa, dastlabki vektor $y^{(0)}$ ni o‘zgartirib boshqa xos vektorlarni topish mumkin.

4-hol. A matritsaning moduli bo‘yicha eng kotta sonlar kompleks yoki modullari bilan o‘zaro yaqin bo‘lhan holni ko‘ramiz. Faraz qilaylik, λ_1 va λ_2 xos sonlar qo‘shma kompleks sonlar bo‘lib, quyidagi shartlar o‘rinli bo‘lsin:

$$|\lambda_1| = |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|.$$

Quyidagi taqribiy tengliklar

$$\begin{cases} y^{(k)} \cong b_1\lambda_1^k x^{(1)} + b_2\lambda_2^k x^{(2)} \\ y^{(k+1)} \cong b_1\lambda_1^{k+1} x^{(1)} + b_2\lambda_2^{k+1} x^{(2)} \\ (y^{(k+2)} \cong b_1\lambda_1^{k+2} x^{(1)} + b_2\lambda_2^{k+2} x^{(2)} \end{cases} \quad (13)$$

mavjudligiga ishonch hosil qilish qiyin emas. Bular orasida

$$y^{(k+2)} - (\lambda_1 + \lambda_2)y^{(k+1)} + \lambda_1\lambda_2 y^{(k)} = 0 \quad (14)$$

chiziqli bog‘lanish o‘rinlidir. Agar hisoblash jarayonida $y^{(k)}, y^{(k+1)}, y^{(k+2)}$ vektorlar orasida

$$y^{(k+2)} + py^{(k+1)} + qy^{(k)} = 0 \quad (15)$$

chiziqli bog‘lanish borligini ko‘rsak, u holda λ_1 va λ_2 lar

$$u^2 + pu + q = 0 \quad (16)$$

kvadrat tenglamani qanoatlantiradi. (16) kvadrat tenglamaning koeffitsiyentlari quyidagi determinantlardan aniqlanadi:

$$\begin{vmatrix} 1 & y_i^{(k)} & y_j^{(k)} \\ U & y_i^{(k+1)} & y_j^{(k+1)} \\ U^2 & y_i^{(k+2)} & y_j^{(k+2)} \end{vmatrix} = 0, i \neq j, i, j = 1,2, \dots, n \quad (17)$$

Demak, (17) dan p va q topiladi, (16)dan esa λ_1 va λ_2 aniqlanadi, so‘ngra (13) dan foydalanib xos vektorlar quyidagicha topiladi:

$$y^{(k+1)} - \lambda_1 y^{(k)} = b_2\lambda_2{}^k(\lambda_2 - \lambda_1)x^{(1)},$$
$$y^{(k+1)} - \lambda_2 y^{(k)} = b_1\lambda_1{}^k(\lambda_1 - \lambda_2)x^{(2)}.$$

Eslatma. Birinchi va ikkinchi hollarda $y^{(0)}$ vektorning iteratsiyalarini topish lozim edi. Shu jarayonni tezlashtirish uchun quyidagicha yo‘l tutiladi:

$$A, A^2, A^4, A^8 \dots A^{2^k}$$

matritsalar ketma-ketligini hosil qilamiz.

Ma'lumki,

$$\sum_{i=1}^{n} \lambda_i = SpA,$$
$$\sum_{i=1}^{n} \lambda_i{}^{2}k = SpA^{2^k}$$

Bundan foydalanib, misol uchun 1-holda

$$\sum_{i=1}^{n} \lambda_i = \mathrm{Sp}A,$$

$$\sum_{i=1}^{n} \lambda_i{}^{2}k = \mathrm{Sp}A^{2^k}.$$

lardan

$$\lambda_1 \cong \frac{SpA^{2^{k+1}}}{SpA^{2^k}}$$

ekanligi kelib chiqadi.

Xulosa: Ushbu maqolada matritsaning moduli bo‘yicha eng katta xos sonini va unga mos keluvchi xos vektorlarni aniqlash masalasi nazariy jihatdan tahlil qilindi. Kuchaytirish usuli asosida iteratsion yondashuvlar yordamida bu qiymatlarni topish bosqichlari batafsil ko‘rib chiqildi. Tadqiqotlar shuni ko‘rsatdiki, moduli bo‘yicha eng katta xos sonni aniqlash ko‘plab amaliy sohalarda jumladan, fizika, muhandislik, iqtisodiyot va kompyuter fanlarida muhim ahamiyatga ega. Berilgan algoritmlar va misollar asosida bu metodning qulayligi va samaradorligi isbotlandi. Kelgusida murakkab matritsalar, katta o‘lchamli tizimlar va parallel hisoblash muhitida ham ushbu usullarni takomillashtirish mumkinligi ta’kidlab o‘tildi.

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