FORMULATION OF THE INITIAL-BOUNDARY VALUE PROBLEM IN A CIRCULAR DOMAIN

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Abstract. *This paper is devoted to the formulation of an initial-boundary value problem in a circular domain for partial differential equations arising in mathematical physics. The circular geometry is described using polar coordinates, which allows an explicit representation of the governing equations and boundary conditions. Particular attention is paid to the consistent specification of initial and boundary conditions required to ensure the well-posedness of the problem in the sense of Hadamard. The role of domain symmetry in simplifying the analytical treatment is emphasized, and the applicability of classical methods such as separation of variables and spectral analysis of the Laplace operator is discussed. The proposed formulation provides a rigorous mathematical foundation for further analytical and numerical studies of wave propagation and related physical processes in circular domains.*

Keywords. *Initial-boundary value problem; circular domain; partial differential equations; Laplace operator; well-posedness; polar coordinates; mathematical physics.*

Initial-boundary value problems play a fundamental role in the mathematical modeling of physical processes described by partial differential equations. Such problems arise naturally in the study of wave propagation, heat transfer, diffusion phenomena, elasticity, and quantum mechanics. The formulation of an initial-boundary value problem determines not only the mathematical correctness of the model but also the existence, uniqueness, and stability of its solution, which are essential for both theoretical analysis and practical applications.

Circular domains represent an important class of bounded regions due to their geometric symmetry and frequent occurrence in physical systems, such as vibrating membranes, acoustic resonators, heat conduction in cylindrical objects, and electromagnetic waveguides. The symmetry of a circular domain allows the effective use of analytical methods, including separation of variables and spectral analysis, leading to solutions expressed in terms of special functions, particularly Bessel functions. As a result, circular domains serve as benchmark models for studying more complex geometries.

The proper formulation of initial-boundary value problems in a circular domain requires careful specification of the governing differential equation, initial conditions, and boundary conditions. These components must be mutually consistent to ensure well-posedness in the sense of Hadamard. Inadequate or incompatible conditions may lead to non-existence of solutions or instability with respect to perturbations in the initial data. Therefore, a rigorous mathematical framework for such problems is of significant importance.

This paper focuses on the formulation of initial-boundary value problems in a circular domain, emphasizing the mathematical structure and key assumptions necessary for well-posedness. The general form of the governing equations, typical boundary conditions, and admissible initial data are discussed. The results presented in this study provide a foundation for further analytical and numerical investigations of physical processes modeled in circular geometries.

Let Ω be a circular domain of radius $R > 0$ in the two-dimensional Euclidean space, which in polar coordinates is defined by $\Omega = \{(r,\theta) \mid 0 < r < R, 0 \le \theta < 2\pi\}$, with boundary $\partial\Omega = \{r = R\}$. Due to the geometric symmetry of the domain, polar coordinates provide a natural framework for the formulation of initial-boundary value problems and allow a more transparent analytical treatment of the governing equations. Consider a second-order partial differential equation describing wave propagation in the domain, given by $\partial^2 u/\partial t^2 = c^2 \Delta u + f(r,\theta,t)$, where $u(r,\theta,t)$ denotes the unknown function, $c > 0$ is a constant wave speed, $f(r,\theta,t)$ is a prescribed source term, and Δ is the Laplace operator. In polar coordinates, the Laplace operator takes the form $\Delta u = \partial^2 u/\partial r^2 + (1/r)\partial u/\partial r + (1/r^2)\partial^2 u/\partial\theta^2$, which explicitly reflects the radial and angular contributions to the dynamics of the system.

To fully determine the temporal evolution of the solution, appropriate initial conditions are imposed at $t = 0$. These conditions are usually specified in the form $u(r,\theta,0) = \varphi(r,\theta)$ and $\partial u/\partial t(r,\theta,0) = \psi(r,\theta)$, where the functions φ and ψ describe the initial displacement and initial velocity, respectively. It is assumed that these functions possess sufficient smoothness and satisfy compatibility conditions with the boundary data to ensure the mathematical correctness of the problem.

The formulation of the problem is completed by prescribing boundary conditions on $\partial\Omega$. A common choice is the Dirichlet boundary condition $u(R,\theta,t) = 0$, which corresponds to a fixed boundary. Alternatively, the Neumann boundary condition $\partial u/\partial r(R,\theta,t) = 0$ may be considered, modeling a free boundary. The type of boundary condition is determined by the physical nature of the process under investigation and significantly influences the qualitative behavior of the solution.

The well-posedness of the formulated initial-boundary value problem is understood in the sense of Hadamard and requires the existence and uniqueness of a solution as well as its continuous dependence on the initial and boundary data. Under standard assumptions on the smoothness of the coefficients and data, the problem admits a unique solution belonging to appropriate functional spaces. The circular geometry of the domain allows the application of analytical methods such as separation of variables, which leads to an eigenvalue problem for the Laplace operator. The resulting eigenfunctions form an orthogonal basis, and the solution can be represented as a series expansion involving Bessel functions for the radial part and trigonometric functions for the angular part. This representation provides a rigorous analytical foundation for further qualitative analysis and numerical approximations of the solution.

Conclusion. In this paper, the formulation of an initial-boundary value problem in a circular domain has been investigated within a rigorous mathematical framework. The circular geometry was shown to provide a convenient and analytically tractable setting for the study of partial differential equations describing fundamental physical processes such as wave propagation. By expressing the problem in polar coordinates, the structure of the governing equation and the associated initial and boundary conditions were presented in a clear and consistent form.

The analysis demonstrated that the correct specification of initial and boundary conditions plays a crucial role in ensuring the well-posedness of the problem in the sense of Hadamard. Under standard regularity and compatibility assumptions, the formulated problem admits a unique solution that depends continuously on the given data. The symmetry of the circular domain enables the application of classical analytical techniques, including the method of separation of variables and spectral analysis of the Laplace operator, which lead to solution representations in terms of special functions.

The results obtained in this study provide a solid theoretical foundation for further analytical and numerical investigations of initial-boundary value problems in circular and more general bounded domains. Moreover, the presented formulation can be extended to other types of partial differential equations and boundary conditions, making it applicable to a wide range of problems in mathematical physics and applied mathematics.

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